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# ON THE ISOTROPIC CONSTANT OF NON-SYMMETRIC CONVEX BODIES

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ABSTRACT

We estimate the isotropic constant of non-symmetric convex bodies and discrete sets in $\mathbb{R}^n$.

## 1. Introduction

The notions of isotropic position and isotropic constant are usually defined and discussed for convex symmetric bodies, see [B1] and [MP]. However, they can be easily adapted to make sense also in the non-symmetric case:

For a convex body $K \subset \mathbb{R}^n$ we define its isotropic constant $L_K$ by

$$nL_K^2|K|^{2/n} = \inf_{\substack{T\in \mathrm{SL}(n)\\ t\in\mathbb{R}^n}} \frac{1}{|K|}\int_K |Tx+t|^2 dx \tag{1}$$

(where $|K|$ stands for the volume of $K$). If this infimum is attained for $T = \mathrm{id}$, $t = 0$ we say that $K$ is in isotropic position.

$L_K$ is an affine invariant by definition and clearly each $K$ has an affine image in isotropic position. For a convex body $K \subset \mathbb{R}^n$ to be in isotropic position (with isotropic constant $L_K$) it is necessary and sufficient that the following two conditions be satisfied:

(2) The centre of mass of $K$ is in the origin.

$$\frac{1}{|K|}\int_K \langle x,\varphi\rangle\langle x,\psi\rangle dx = L_K^2\langle\varphi,\psi\rangle|K|^{2/n} \quad \forall\varphi,\psi\in\mathbb{R}^n. \tag{3}$$

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As (2) and (3) determine the position of $K$ uniquely, it is enough to show they are valid in the isotropic position.

Indeed, if the infimum in (1) is attained for $T = \mathrm{id}$, $t = 0$ then for any $\varphi \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}$ we have

$$\frac{1}{|K|}\int_K |x|^2 dx \leq \frac{1}{|K|}\int_K |x+\varepsilon\varphi|^2 dx = \frac{1}{|K|}\int_K |x|^2 dx + 2\varepsilon\frac{1}{|K|}\int_K \langle x,\varphi\rangle dx + O(\varepsilon^2)$$

from which it follows that $\frac{1}{|K|}\int_K \langle x,\varphi\rangle dx = 0$ for all $\varphi \in \mathbb{R}^n$, hence (2).

Similarly, for any $S \in M_{n\times n}(\mathbb{R})$ and $\varepsilon \in \mathbb{R}$ we have

$$\begin{aligned} nL_K^2|K|^{2/n}\left(1+\frac{2}{n}\varepsilon \operatorname{tr} S + O(\varepsilon^2)\right) &= nL_K^2|K|^{2/n}\det(I+\varepsilon S)^{2/n} \\ &\leq \frac{1}{|K|}\int_K |x+\varepsilon Sx|^2 dx = \frac{1}{|K|}\int_K |x|^2 dx + 2\varepsilon\frac{1}{|K|}\int_K \langle x, Sx\rangle dx + O(\varepsilon^2) \\ &= nL_K^2|K|^{2/n} + 2\varepsilon\frac{1}{|K|}\int_K \langle x, Sx\rangle dx + O(\varepsilon^2) \end{aligned}$$

from which it follows that $\frac{1}{|K|}\int_K \langle x, Sx\rangle dx = L_K^2|K|^{2/n}\operatorname{tr} S$ and, taking $S = \varphi \otimes \psi$, we get (3).

The problem, initiated by Bourgain, whether $\sup_n \sup_{K\subset\mathbb{R}^n} L_K < \infty$, is still open even in the symmetric case and it was investigated in many particular cases (see [D] for discussion and other references). Keith Ball [B2] believes the symmetric convex body with maximal isotropic constant to be the cube in any dimension for which it is $1/\sqrt{12}$. It is interesting to notice that for the $n$-simplex we already get $\sim 1/e > 1/\sqrt{12}$. Although estimating $L_K$ is generally a difficult problem, showing $L_K \lesssim \sqrt{n}$ for a convex symmetric body $K \subset \mathbb{R}^n$ is quite easy and well-known. We need only observe that, for any $K \subset \mathbb{R}^n$, $L_K \lesssim d(K,D)$ where $D$ is the standard Euclidean ball in $\mathbb{R}^n$ and

$$d(K,D) = \inf_{\substack{T\in SL(n)\\ t\in\mathbb{R}^n}}\left\{\frac{\beta}{\alpha} \mid \alpha D \subset TK + t \subset \beta D\right\}.$$

Let us take $T, t$ which achieve this infimum and then

$$nL_K^2|K|^{2/n} \leq \frac{1}{|K|}\int_K |Tx+t|^2 dx \leq \beta^2,$$

$$\frac{\alpha^2}{n} \sim |\alpha D|^{2/n} \leq |TK+t|^{2/n} = |K|^{2/n},$$

hence $L_K \lesssim \beta/\alpha = d(K,D)$ as claimed.

To conclude the estimate of $L_K$ we use John's fundamental result which ensures us $d(K, D) \lesssim \sqrt{n}$ in the symmetric case. Unfortunately, for non-symmetric bodies this only yields $L_K \lesssim n$. However, we will show here that

THEOREM 1: *$L_K \lesssim \sqrt{n}$ for any convex body $K \subset \mathbb{R}^n$.*

In order to prove this, we first introduce the notion of isotropy for yet another case—discrete sets of points in $\mathbb{R}^n$. We give a sharp estimate of the isotropic constant in that case and end by using a somewhat surprising connection between the two problems.

It should be noted here that after reading this note, A. Pajor found a proof using an existing approach for estimating the isotropic constant—through volume ratio. However, this way recruits two quite difficult results—Bourgain and Milman's inverse Santaló inequality and Ball's determination of maximal volume ratio, whereas the argument given below is totally elementary and, we believe, may be used in other situations.

## 2. Isotropic capacity of finite sets in $\mathbb{R}^n$

*Definition:* A system $\{(v_i, \lambda_i)\}_{i=1}^N$ of points $v_1, v_2, \ldots, v_N \in \mathbb{R}^n$ together with weights $\lambda_1, \ldots, \lambda_N \geq 0$, $\sum_{i=1}^N \lambda_i = 1$ is said to be in isotropic position with isotropic constant $L$ if the following two conditions are satisfied:

$$(2') \qquad \sum_{i=1}^N \lambda_i v_i = 0,$$

$$(3') \qquad \sum_{i=1}^N \lambda_i \langle v_i, \varphi\rangle\langle v_i, \psi\rangle = L^2 |\operatorname{conv}\{v_i\}_{i=1}^N|^{2/n} \langle \varphi, \psi\rangle \quad \forall \varphi, \psi \in \mathbb{R}^n.$$

PROPOSITION 2: *In the above setting $L \lesssim \sqrt{n}$ and this estimate cannot be generally improved.*

*Proof:* Let $A$ be the $n \times N$ matrix whose $i$-th column is $\sqrt{\lambda_i} v_i$. Then the isotropicy of the system $(v_i, \lambda_i)$ simply means that $AA^t = L^2 |\operatorname{conv}\{v_i\}_{i=1}^N|^{2/n} \cdot I_n$.

So, by the Cauchy-Binet identity

$$\begin{aligned} L^{2n}|\operatorname{conv}\{v_i\}_{i=1}^N|^2 &= \det(L^2|\operatorname{conv}\{v_i\}_{i=1}^N|^{2/n}\cdot I_n) = \det(A\cdot A^t) \\ &= \sum_{1\le i_1<i_2<\cdots<i_n\le N} \det(A_{i_1 i_2\cdots i_n}A^t_{i_1 i_2\cdots i_n}) \\ &= \sum_{1\le i_1<i_2<\cdots<i_n\le N} \det(A_{i_1\cdots i_n})^2 \\ &= \sum_{1\le i_1<i_2\cdots<i_n\le N} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}\det(v_{i_1},v_{i_2}\cdots v_{i_n})^2 \ . \end{aligned}$$

But $\det(v_{i_1}v_{i_2}\cdots v_{i_n}) = n!|\operatorname{conv}\{0,v_{i_1},\ldots,v_{i_n}\}| \le n!|\operatorname{conv}\{v_i\}_{i=1}^N|$, so

$$L^{2n} \le \left(\sum_{1\le i_1<i_2<\cdots<i_n\le N} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}\right)\cdot(n!)^2|\operatorname{conv}\{v_i\}_{i=1}^N|^2 \ .$$

Also by Maclaurin's inequality

$$\sum_{1\le i_1<i_2<\cdots<i_n\le N} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} \le \binom{N}{n}\left[\frac{\sum_{i=1}^N\lambda_i}{N}\right]^n = \binom{N}{n}\frac{1}{N^n} \le \frac{1}{n!}.$$

Hence $L \le \sqrt[2n]{n!} \sim \sqrt{n}$. For the set of $2n$ points $\{\pm e_i\}_{i=1}^n$ with equal weights $\lambda_i = 1/2n$ we get $L = \sqrt[n]{n!}/2\sqrt{n} \sim \sqrt{n}$. ∎

LEMMA 3: *Let $K = \bigcup_{i=1}^N \Delta_i$ (i.e. $\operatorname{int}\Delta_i \cap \operatorname{int}\Delta_j = \emptyset$ for $i \ne j$) where $\Delta_i = \operatorname{conv}\{0, v_i^1,\ldots,v_i^n\}$ and let $w_i = \frac{1}{n}\sum_{j=1}^n v_i^j$. Then if $K$ is in isotropic position the system $\left\{(v_i^j, \frac{|\Delta_i|}{n(n+1)|K|})\right\}_{\substack{i=1,\ldots,N\\ j=1,\ldots,n}} \cup \left\{(w_i, \frac{n|\Delta_i|}{(n+1)|K|})\right\}_{i=1,\ldots,N}$ is also in isotropic position with isotropic constant $\sqrt{\frac{n+2}{n}}L_K$.*

*Proof:* The centre of mass of each simplex $\Delta_i$ is

$$\frac{0+v_i^1+\cdots+v_i^n}{n+1} = \frac{n}{n+1}w_i + \frac{1}{n(n+1)}v_i^1 + \cdots + \frac{1}{n(n+1)}v_i^n,$$

hence the centre of mass of $K$ is just

$$\sum_{i=1}^N\sum_{j=1}^n \frac{|\Delta_i|}{n(n+1)|K|}v_i^j + \sum_{i=1}^N \frac{n|\Delta_i|}{(n+1)|K|}w_i$$

and (2) implies (2′).

For any $i = 1, \dots, N$ let $T_i\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T_i e_j = v_i^j$, and let $\Delta = \operatorname{conv}\{0, e_1, \dots, e_n\}$. Then $\forall \varphi, \psi$

$$
\begin{aligned}
&\frac{1}{|\Delta_i|}\int_{\Delta_i} \langle y,\varphi\rangle\langle y,\psi\rangle dy = \frac{1}{|\Delta_i|}\int_{\Delta} \langle T_i x,\varphi\rangle\langle T_i x,\psi\rangle |\det T_i| dx \\
&= n! \underset{\substack{0\le x_1,\dots,x_n\\ x_1+\dots+x_n\le 1}}{\int\int\cdots\int} \left(\sum_{j=1}^n x_j \langle v_i^j,\varphi\rangle\right)\left(\sum_{j=1}^n x_j \langle v_i^j,\psi\rangle\right) dx_1 dx_2\cdots dx_n \\
&= n!\Bigg[\sum_{j=1}^n \langle v_i^j,\varphi\rangle\langle v_i^j,\psi\rangle \underset{\substack{0\le x_1,\dots,x_n\\ x_1+\dots+x_n\le 1}}{\int\int\cdots\int} x_j^2 dx_1\cdots dx_n + \\
&\qquad + \sum_{\substack{1\le j,k\le n\\ j\ne k}} \langle v_i^j,\varphi\rangle\langle v_i^k,\psi\rangle \underset{\substack{0\le x_1,\dots,x_n\\ x_1+\dots+x_n\le 1}}{\int\int\cdots\int} x_j x_k dx_1\cdots dx_n\Bigg] \\
&= n!\left[\sum_{j=1}^n \langle v_i^j,\varphi\rangle\langle v_i^j,\psi\rangle \frac{2}{(n+2)!} + \sum_{\substack{1\le j,k\le n\\ j\ne k}} \langle v_i^j,\varphi\rangle\langle v_i^k,\psi\rangle \frac{1}{(n+2)!}\right] \\
&= \frac{1}{(n+1)(n+2)}\left[\sum_{j=1}^n \langle v_i^j,\varphi\rangle\langle v_i^j,\psi\rangle + \left\langle \sum_{j=1}^n v_i^j,\varphi\right\rangle\left\langle \sum_{j=1}^n v_i^j,\psi\right\rangle\right] \\
&= \frac{1}{(n+1)(n+2)}\left[\sum_{j=1}^n \langle v_i^j,\varphi\rangle\langle v_i^j,\psi\rangle + n^2\langle w_i,\varphi\rangle\langle w_i,\psi\rangle\right].
\end{aligned}
$$

Summing up we get:

$$
L_K^2|K|^{2/n}\langle\varphi,\psi\rangle = \frac{1}{|K|}\int_K \langle x,\varphi\rangle\langle x,\psi\rangle dx
$$

$$
\begin{aligned}
&= \sum_{i=1}^N \frac{|\Delta_i|}{|K|}\left(\frac{1}{|\Delta_i|}\int_{\Delta_i} \langle x,\varphi\rangle\langle x,\psi\rangle dx\right) \\
&= \sum_{i=1}^N \frac{|\Delta_i|}{|K|(n+1)(n+2)}\left[\sum_{j=1}^n \langle v_i^j,\varphi\rangle\langle v_i^j,\psi\rangle + n^2\langle w_i,\varphi\rangle\langle w_i,\psi\rangle\right] \\
&= \frac{n}{n+2}\Bigg(\sum_{i=1}^N\sum_{j=1}^n \frac{|\Delta_i|}{|K|n(n+1)}\langle v_i^j,\varphi\rangle\langle v_i^j,\psi\rangle \\
&\qquad + \sum_{i=1}^N \frac{|\Delta_i| n}{|K|(n+1)}\langle w_i,\varphi\rangle\langle w_i,\psi\rangle\Bigg)
\end{aligned}
$$

and as $|\operatorname{conv}(\{v_i^j\}_{\substack{i=1,\dots,N\\ j=1,\dots,N}} \cup \{w_i\}_{i=1,\dots,N})| = |\operatorname{conv}\{v_i^j\}_{\substack{i=1,\dots,N\\ j=1,\dots,N}}| = |K|$ we get that (3) implies (3′) with $L = \sqrt{(n+2)/n}L_K$ as claimed. ∎

*Proof of Theorem 1:* It is enough to prove the result for polytopes. Let $K$ be a polytope in isotropic position, so $0 \in K$ and we have a decomposition $K = \bigcup_{i=1}^N \Delta_i$ where each $\Delta_i = \operatorname{conv}\{0, v_i^1, \dots, v_i^n\}$. By Lemma 3 the points $\{v_i^j\}_{i=1}^N{}_{j=1}^n \cup \left\{w_i = \frac{\sum_{j=1}^n v_i^j}{n}\right\}_{i=1}^N$ with appropriate weights are in isotropic position with isotropic constant $\sqrt{\frac{n+2}{n}}L_K$. Now, by Proposition 2 this constant is $\lesssim \sqrt{n}$, hence $L_K \lesssim \sqrt{n}$. ∎

*Remark:* Examining closer the proofs of Proposition 2 and Theorem 1, we see that actually the following stronger estimate holds for any convex $K \subset \mathbb{R}^n$:

$$L_K \lesssim \sqrt{n} \cdot \sup_{v_0, v_1, \dots, v_n \in K} \left( \frac{|\operatorname{conv}\{v_i\}_{i=0}^n|}{|K|} \right)^{1/n}.$$

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